Control Systems I

State Space

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State-Space Control Design

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

This is ***equivalent*** to the transfer function form studied previously.

Design a controller of the form $u = \kappa(x)$

Why do we need another control design procedure?

- More general
  - Easily extends to multiple inputs and outputs
  - Can capture nonlinear systems in this form too
- Phase plane / geometry
  - System safely and physical limitations can often be described as geometric constraints in the motion of the state through the phase space
- Optimal control
  - Complex time-domain objectives can be easily specified in terms of the states

State-Space Design Procedure

1 State-Feedback Design

Assume that the ***state is measured***, and design a ***static*** control law $u = Kx$

$$\dot{x} = Ax + BKx$$

Problem : We can't measure x!

2 State Observer

Design a dynamic system to ***estimate the state***

$$\dot{\hat{x}} = L\hat{x} + My + Nu$$

Design L, M and N so that $\hat{x} \sim x$

3 Combine controller and observer to provide a single, dynamic control law.

4 Add reference tracking.

Separation principle tells us that independent design of these elements is *optimal*.

Recall: State-Variable Form

Recall: Quick review on linear state-space

Write the following ODE in state-space form

$$\ddot{\theta} + b\dot{\theta} + c\theta = du$$

Recall: Quick review on linear state-space

Write the following ODE in state-space form

$$\ddot{\theta} + b\dot{\theta} + c\theta = du$$

Introduce state variables

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

Take derivates

$$\begin{aligned}\dot{x}_1 &= \dot{\theta} = x_2 \\ \dot{x}_2 &= \ddot{\theta} = -b\dot{\theta} - c\theta + du \\ &= -bx_2 - cx_1 + du\end{aligned}$$

Write out the state-space equations

$$\begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} = \begin{bmatrix}0 & 1 \\ -c & -b\end{bmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} + \begin{bmatrix}0 \\ d\end{bmatrix} u$$

Linearization

In a later lecture, we will cover multiple ways of generating linear state-space equations

- Linearization of nonlinear ODEs
- Static Feedback Linearization
- Nonlinear inversion
- Experimental linearization

For now: Recall your system dynamique notes on linear state-space modeling

Dynamic Response

State-Space $\rightarrow$ Transfer Function

Compute transfer function

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

State-Space $\rightarrow$ Transfer Function

Compute transfer function

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Take Laplace transform

$$sX(s) - x(0) = AX(s) + BU(s)$$

Solve for $X(s)$

$$X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0)$$

The output is

$$\begin{aligned} Y(s) &= CX(s) + DU(s) \\ &= (C(sI - A)^{-1}B + D)U(s) + C(sI - A)^{-1}x(0) \end{aligned}$$

Assuming zero initial conditions gives the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Example

Compute transfer function

$$\dot{x} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

Example

Compute transfer function

$$\dot{x} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s+7 & 12 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{(s+7)s+12} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s & -12 \\ 1 & s+7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{s+2}{s^2+7s+12} \end{aligned}$$

Poles

Poles are complex frequencies the system will respond at without a forcing function

Consider the system without the forcing function u

$$\dot{x} = Ax$$

assume an initial state $x(0) = x_0$.
p is a pole of the system if the system evolves as $x(t) = e^{pt}x_0$.

From the system dynamics

$$\dot{x}(t) = pe^{pt}x_0 = Ax(t) = Ae^{pt}x_0$$

and therefore p is a pole if

$$Ax_0 = px_0$$

or if p is an ***eigenvalue*** of A.

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and therefore p is a pole if

$$Ax_0 = px_0$$

or if p is an ***eigenvalue*** of A.

The poles are the solutions of the ***characteristic equation***

$$\det(sI - A) = 0$$

Example

Compute the poles

$$\dot{x} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Example

Compute the poles

$$\dot{x} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Poles are given by the characteristic equation

$$\begin{aligned} \det(sI - A) &= 0 \\ &= \left| \begin{bmatrix} s+7 & 12 \\ -1 & s \end{bmatrix} \right| \\ &= (s+7)s + 12 = s^2 + 7s + 12 \end{aligned}$$

Poles are

$$p = \frac{-7 \pm \sqrt{49 - 4 \cdot 12}}{2} = -4, -3$$

One can see that this is the same as the poles computed from the transfer function

$$\frac{s+2}{s^2 + 7s + 12} = \frac{s+2}{(s+4)(s+3)}$$

Zeros

Zeros are generalized frequencies at which the system will not respond to an input

z is a zero if $u(t) = u_0 e^{zt} \to y(t) = 0$

Take $u(t) = u_0 e^{zt}$, then $x(t) = x_0 e^{zt}$

$$\dot{x} = z e^{zt} x_0 = A e^{zt} x_0 + B u_0 e^{zt} \qquad \Leftrightarrow \qquad \begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

Combining this with $y = Cx + Du = C e^{zt} x_0 + D u_0 e^{zt}$ gives

$$\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The zeros of the system are given by the expression

$$\det \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = 0$$

Example

Compute the Zeros

$$\dot{x} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

Example

Compute the Zeros

$$\dot{x} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

$$\det \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = \det \begin{bmatrix} z+7 & 12 & -1 \\ -1 & z & 0 \\ 1 & 2 & 0 \end{bmatrix}$$
$$= 2 + z$$

There is one zero at $z = -2$

Compare to the transfer function, and one sees we get the right result

$$G(s) = \frac{s+2}{s^2 + 7s + 12}$$

Canonical Forms

How to Choose a State Representation

Two state representations can have ***exactly*** the same input-output behaviour

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad == \qquad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} + \bar{D}u \end{aligned}$$

We choose the state representation depending on what we're trying to achieve.

- Control canonical form $\leftarrow$ Used to design controllers
- Modal canonical form $\leftarrow$ Used to analyse oscillation modes
- Observer canonical form $\leftarrow$ Used to design observers

Control Canonical Form

Goal: Form that allows for simple modification of the system dynamics.

Consider the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12}$$

Introduce an intermediary variable x_2[1]

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{X_2(s)} \cdot \frac{X_2(s)}{U(s)} = \frac{s+2}{1} \cdot \frac{1}{s^2+7s+12}$$

State to output   Input to state

[1] x_2 because this is a second order system

Input $\rightarrow$ State

$$\frac{X_2(s)}{U(s)} = \frac{1}{s^2 + 7s + 12}$$

Write the dynamic equation relating x_2 to the input u

$$u = \ddot{x}_2 + 7\dot{x}_2 + 12x_2 \tag{1}$$

Introduce new state variable for the derivative

$$\dot{x}_2 = x_1$$

Re-write (1) in terms of the derivative of the state x_1

$$\dot{x}_1 = u - 7x_1 + 12x_2$$

Input to state equations

$$\begin{aligned} \dot{x}_1 &= u - 7x_1 + 12x_2 \\ \dot{x}_2 &= x_1 \end{aligned}$$

State to output

$$\frac{Y(s)}{X_2(s)} = \frac{s+2}{1}$$

Convert to the time domain

$$y = \dot{x}_2 + x_2 = x_1 + x_2$$

where we used the definition $\dot{x}_2 = x_1$ from the previous slide

Put it all together to get the control canonical form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -7 & 12 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Consider the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

The control canonical form is

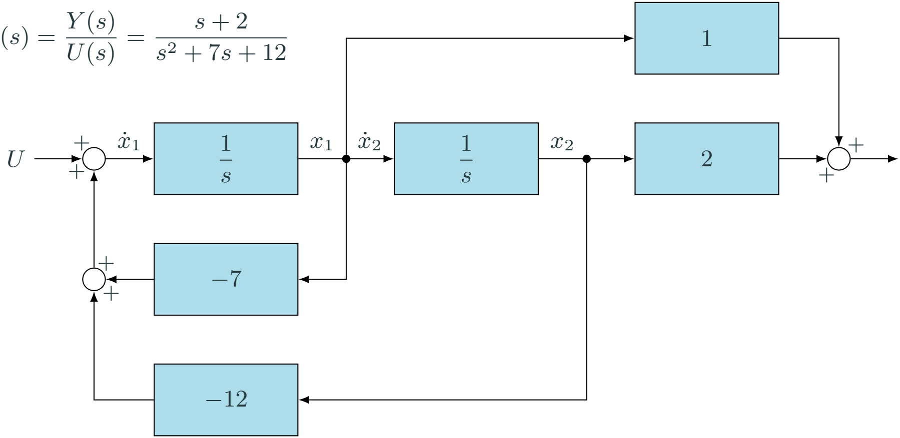
$$A = \begin{bmatrix} -a_1 & -a_2 & \dots & \dots & -a_n \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} b_1 & b_2 & \dots & \dots & b_n \end{bmatrix} \qquad D = 0$$

It is also possible to convert to control canonical form if b_0 is not zero (i.e., if there are n zeros, rather than $n-1$). In this case, the expression for C is slightly more complex and D is non-zero.

Block Diagram - Control Canonical Form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12}$$



Control canonical form block diagram

- All dynamic blocks are integrators
- Output and input are weighted sums of the states

State Transformations

Consider the state equations

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

This representation is not unique. Consider a change of variables given by the *nonsingular* matrix T

$$x = Tz$$

The *same* dynamic system expressed in terms of the state z is now

$$\dot{x} = T\dot{z} = ATz + Bu$$
$$\dot{z} = T^{-1}ATz + T^{-1}Bu$$
$$y = CTz + Du$$

We get a new state representation for the *same dynamic system*

$$\dot{z} = \bar{A}z + \bar{B}u \qquad \bar{A} = T^{-1}AT \quad \bar{B} = T^{-1}B$$
$$y = \bar{C}z + \bar{D}u \qquad \bar{C} = T^{-1}C \quad \bar{D} = D$$

State Transformation to Control Canonical Form

Goal: Convert from any representation to control canonical form.

$$\overbrace{\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}}^{\text{Any representation}} \qquad \to T \to \qquad \overbrace{\begin{aligned}\dot{z} &= \bar{A}z + \bar{B}u \\ y &= \bar{C}z + \bar{D}u\end{aligned}}^{\text{Control canonical form}}$$

Target structure

$$\bar{A} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \bar{B} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

State Transformation to Control Canonical Form

The relationship between the state matrices is

$$\bar{A}T^{-1} = T^{-1}A$$

Let the rows of T^{-1} be t_1, t_2 and t_3, and let $\bar{A}$ be in control canonical form

$$\begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} t_1 A \\ t_2 A \\ t_3 A \end{bmatrix}$$

We can write the transform matrix T in terms of its last row t_3

$$\begin{aligned} t_2 &= t_3 A \\ t_1 &= t_2 A = t_3 A^2 \end{aligned}$$

State Transformation to Control Canonical Form

Relationship between the input matrices is

$$T^{-1}B = \begin{pmatrix} t_1 B \\ t_2 B \\ t_3 B \end{pmatrix} = \bar{B} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Combining with the equations from the previous slide gives an expression for t_3

$$\begin{aligned} t_2 &= t_3 A \\ t_1 &= t_3 A^2 \end{aligned} \qquad \rightarrow \qquad t_3 \underbrace{\begin{bmatrix} B & AB & A^2 B \end{bmatrix}}_{\text{Controllability matrix } \mathcal{C}} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

Finally, the last row of the transformation is given by

$$t_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathcal{C}^{-1}$$

State Transformation to Control Canonical Form

General procedure

1. Form the controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

2. Compute the last row of the inverse

$$t_n = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix} \mathcal{C}^{-1}$$

3. Construct the transformation matrix

$$T^{-1} = \begin{bmatrix} t_n A^{n-1} \\ t_n A^{n-2} \\ \vdots \\ t_n \end{bmatrix}$$

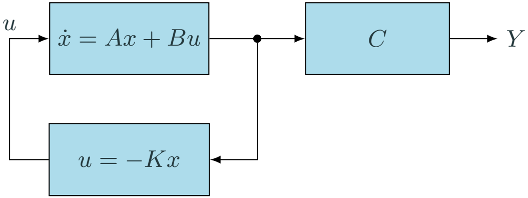
Note : The system can only be put in controllable form if $\mathcal{C}$ is full rank

Control Law Design: Full State Feedback

Control Law Design - Full State Feedback

Try a *static linear control law*

$$u = -Kx = -\begin{bmatrix} K_1 & K_2 & \dots & K_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$



With such a controller, we can place n poles (i.e., all of them).

Full State Feedback

Goal: Place the poles of the closed-loop system at the given locations

$$s = s_1, s_2, \ldots, s_n$$

The closed-loop dynamics are

$$\dot{x} = Ax - BKx$$

with the poles given by the characteristic equation

$$\det(sI - (A - BK)) = c(s; K) \quad \leftarrow \quad \text{Polynomial linearly parameterized by } K$$

Target characteristic equation is

$$(s - s_1)(s - s_2)\cdots(s - s_n) = \alpha_c(s) \quad \leftarrow \quad \text{Polynomial}$$

Idea: Equate coefficients of $c(s; K) = \alpha_c(s)$ in order to choose K.

Example

Control Law for a Pendulum

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Design a linear state-feedback controller to place both closed-loop poles to $-2\omega_0$

i.e., double the natural frequency and increase damping ratio from 0 to 1.

Example

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i.e., double the natural frequency and increase damping ratio from 0 to 1.

Target characteristic equation:

$$\alpha_c(s) = (s + 2\omega_0)^2 = s^2 + 4\omega_0 s + 4\omega_0^2$$

Parameterized characteristic equation:

$$c(s; K) = \det(sI - (A - BK)) = \det \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} \right\}$$

$$= s^2 + K_2 s + \omega_0^2 + K_1$$

Controller is

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} 3\omega_0^2 & 4\omega_0 \end{bmatrix}$$

Pole Placement in Control Canonical Form

Consider pole placement in control canonical form

$$A = \begin{bmatrix} -a_1 & -a_2 & \dots & \dots & -a_n \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad K = \begin{bmatrix} K_1 & K_2 & \dots & K_n \end{bmatrix}$$

The ***upper companion form*** matrix gives the closed-loop dynamics

$$A - BK = \begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & \dots & \dots & -a_n - K_n \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$c(s; K) = s^n + (a_1 + K_1)s^{n-1} + (a_2 + K_2)s^{n-2} + \cdots + (a_n + K_n)$$

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$$c(s;K) = s^n + (a_1 + K_1)s^{n-1} + (a_2 + K_2)s^{n-2} + \cdots + (a_n + K_n)$$

If the target characteristic equation is given by

$$\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$

The control law is

$$K = \begin{bmatrix} -a_1 + \alpha_1 & -a_2 + \alpha_2 & \cdots & -a_n + \alpha_n \end{bmatrix}$$

Pole Placement in Control Canonical Form

Procedure to place poles at desired locations $\{s_i\}$ given dynamic system (A, B)

1. Compute transformation matrix T to convert to control canonical form (A_c, B_c)
2. Compute control law K_c to place poles at $\{s_i\}$ for (A_c, B_c)
3. Convert control gain back to original state $K = K_c T^{-1}$

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This process is written more succinctly as Ackermann's formula

Ackermann's Formula

Goal Choose controller gain K for the system (A, B) so that the closed-loop system $\dot{x} = (A - BK)x$ has the characteristic equation $\alpha(s)$

$$K = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix} \mathcal{C}^{-1} \alpha(A)$$

where $\alpha(A)$ is the desired characteristic equation evaluated at the matrix A

$$\alpha(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \dots + \alpha_n$$

Example

Compute full-state linear controller such that the closed-loop poles are -6 and -5 for the following system.

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Example

Compute full-state linear controller such that the closed-loop poles are -6 and -5 for the following system.

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Target characteristic equation

$$\alpha(s) = (s+6)(s+5) = s^2 + 11s + 30$$

Ackermann's formula

$$\begin{aligned} K &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{C}^{-1} \alpha(A) \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} B & AB \end{bmatrix}^{-1} (A^2 + 11A + 30I) \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^2 + 11 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + 30 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 14 & 57 \end{bmatrix} \end{aligned}$$

Pole Placement - Summary

A ***static linear controller*** $u = -Kx$ can place the closed-loop poles arbitrarily

Required condition: Controllability matrix $\mathcal{C}$ must be invertible.

Controllability

The Problem

Consider the two different state-space models, and their transfer functions

$$\dot{x} = -2x + 2u$$
$$y = 3x$$
$$\Downarrow$$

$$\dot{z} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} z + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 3 & 2 \end{bmatrix}$$
$$\Downarrow$$

$$\begin{aligned} G(s) &= C(sI - A)^{-1} B \\ &= 3(s+2)^{-1} 2 \\ &= \frac{6}{s+2} \end{aligned}$$

$$\begin{aligned} G(s) &= C(sI - A)^{-1} B \\ &= \begin{bmatrix} 3 & 2 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{s+2} & \frac{2}{s+1} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{6}{s+2} \end{aligned}$$

- The effect of the input on the output is the same in both cases!
- While state z_2 impacts the output, we cannot influence it via the input
- However, noise may well drive z_2

Controllability is a function of the state-space representation

Controllability

Controllability

An LTI system is controllable if, for every $x^\star$ and every $T > 0$, there exists an input function $u(t)$, $0 < t \leq T$, such that the state goes from $x(0) = 0$ to $x(T) = x^\star$.

There exists an input that can move the system from any state to any other state in finite time.

Note that this doesn't mean that the system can be held in that state.

Controllability Test

The LTI system (A, B) is controllable if and only if

$$\operatorname{rank} \mathcal{C} = \operatorname{rank} \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = n$$

where $A \in \mathbb{R}^{n \times n}$

Note that we can place the poles of the closed-loop system if and only if the system is controllable, since we must invert the controllability matrix.

State Transformations and Controllability

Question: Does a state transformation impact the controllability of the system?

Consider a system defined by the matrices (A, B), and the system $(\bar{A}, \bar{B})$ transformed by the invertible matrix T.

$$\begin{aligned}
\mathcal{C}_x &= \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \\
\mathcal{C}_z &= \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \dots & \bar{A}^{n-1}\bar{B} \end{bmatrix} \\
&= \begin{bmatrix} T^{-1}B & T^{-1}ATT^{-1}B & \dots & T^{-1}A^{n-1}TT^{-1}B \end{bmatrix} \\
&= T^{-1}\mathcal{C}_x
\end{aligned}$$

$\therefore \mathcal{C}_z$ is nonsingular if and only if $\mathcal{C}_x$ is

State transformations do not impact controllability

Impact of Controllability on State Gain

Compute a linear state feedback controller to place the closed-loop poles at the roots of $s^2 + 2\zeta\omega_n s + \omega_n^2$

$$A = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -z_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

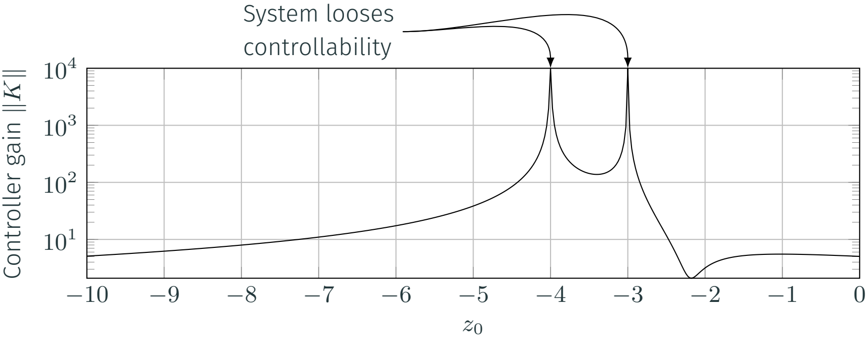
Compute the closed-loop characteristic equation

$$\begin{aligned} \det(sI - (A - BK)) &= s^2 + (K_1 - K_2 z_0 + 7)s - 12K_2 - K_1 z_0 - 7K_2 z_0 + 12 \\ &= s^2 + 2\zeta\omega_n s + \omega_n^2 \end{aligned}$$

$$K_1 = \frac{z_0(14\zeta\omega_n - 37 - \omega_n^2) + 12(2\zeta\omega_n - 7)}{(z_0+3)(z_0+4)} \qquad K_2 = \frac{z_0(7 - 2\zeta\omega_n) + 12 - \omega_n^2}{(z_0+3)(z_0+4)}$$

Take $\zeta = 0.5$, $\omega_n = 2$, and we get the controller

$$K = \frac{1}{(z_0+3)(z_0+4)} \begin{bmatrix} -27z_0 & 5z_0+8 \end{bmatrix}$$



Transfer function of open-loop system is $\dfrac{s-z_0}{(s+4)(s+3)}$

Zero almost cancels one of the poles $\rightarrow$ Higher gain is required to compensate

Assume that the transfer function has distinct real poles p_i[2]

$$
\begin{aligned}
G(s) &= \frac{N(s)}{(s-p_1)(s-p_2)\dots(s-p_n)} \\
&= \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n}
\end{aligned}
$$

Define a set of first-order systems, each with their own state

$$
\begin{aligned}
\frac{X_1}{U(s)} &= \frac{r_1}{s-p_1} & \rightarrow & & \dot{x}_1 &= p_1 x_1 + r_1 u \\
\frac{X_2}{U(s)} &= \frac{r_2}{s-p_2} & \rightarrow & & \dot{x}_2 &= p_2 x_2 + r_2 u \\
&\vdots \\
\frac{X_n}{U(s)} &= \frac{r_n}{s-p_n} & \rightarrow & & \dot{x}_n &= p_n x_n + r_n u
\end{aligned}
$$

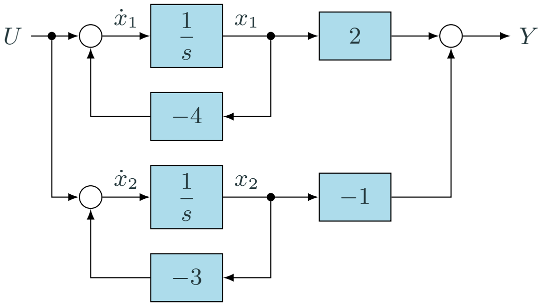
[2]This extends to repeated and complex poles as well, but the resulting A-matrix is no longer diagonal.

Modal Canonical Form

Modal Canonical Form

$$\dot{x} = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{bmatrix} x + \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} r_1 & \dots & r_n \end{bmatrix} x$$



$$G(s) = \frac{s+2}{s^2+7s+12} = \frac{2}{s+4} + \frac{-1}{s+3}$$

$$A = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -1 \end{bmatrix} \qquad D = 0$$

Transformation to Modal Form

Compute the modal form of the system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Compute eigenvalue decomposition of $A = T\Lambda T^{-1}$ (assuming A is diagonalizable)

Apply the state transformation $x = Tz$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu = \Lambda z + T^{-1}Bu$$
$$y = CTz + Du$$

Note that if row i of $T^{-1}B$ is zero, then the input cannot impact mode i, and the model is uncontrollable.

Example

Compute the modal form of the model

$$A = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

$$A = T \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix} T^{-1} \qquad T = \begin{bmatrix} -0.3162 & -0.2425 \\ -0.9487 & -0.9701 \end{bmatrix}$$

Example

Compute the modal form of the model

$$A = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

$$A = T \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix} T^{-1} \qquad T = \begin{bmatrix} -0.3162 & -0.2425 \\ -0.9487 & -0.9701 \end{bmatrix}$$

Modal form

$$\dot{z} = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix} z + \begin{bmatrix} -3.1623 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} -0.3162 & -0.2425 \end{bmatrix} z$$

Where we see from B, that the input has no effect on the second mode z_2.

Reference Tracking

Reference Tracking

If the state input pair (x_{ss}, u_{ss}) satisfies the conditions

$$\begin{aligned} 0 &= Ax_{ss} + Bu_{ss} && \leftarrow && \text{Steady-state} \\ r &= Cx_{ss} + Du_{ss} && \leftarrow && \text{Output equal to the reference} \end{aligned}$$

and the control law $u = u_{ss} - K(x - x_{ss})$ is applied, then

$$\lim_{t \to \infty} y(t) = r$$

Apply control law $u = u_{ss} - K(x - x_{ss})$

$$\begin{aligned} \dot{x} &= Ax + Bu_{ss} - BK(x - x_{ss}) \\ \dot{x} - \dot{x}_{ss} &= Ax + Bu_{ss} - BK(x - x_{ss}) - Ax_{ss} - Bu_{ss} && \text{Add zero to both sides} \\ \frac{d(x - x_{ss})}{dt} &= (A - BK)(x - x_{ss}) \end{aligned}$$

The matrix $(A - BK)$ has all eigenvalues in the negative half space. Therefore x will converge to x_{ss}, and u to u_{ss}.

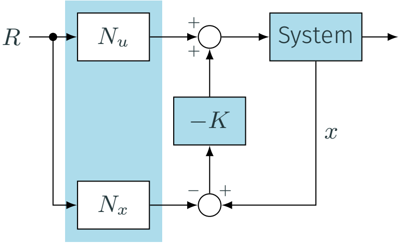
Parameterization of the Target State and Input

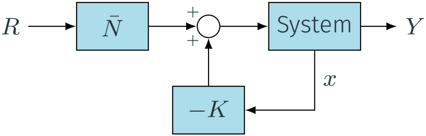
Parameterize the steady-state as a function of r

$$\begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} r = \begin{bmatrix} N_x \\ N_u \end{bmatrix} r$$

The controller is now

$$\begin{aligned} u &= u_{ss} - K(x - x_{ss}) = N_u r - Kx + KN_x r \\ &= -Kx + (N_u + KN_x)r = -Kx + \bar{N}r \end{aligned}$$





Example

Compute full-state linear controller such that the closed-loop poles are -6 and -5 and to track references for the following system.

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Example

Compute full-state linear controller such that the closed-loop poles are -6 and -5 and to track references for the following system.

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

We previously computed the pole placement control law

$$K = \begin{bmatrix} 14 & 57 \end{bmatrix}$$

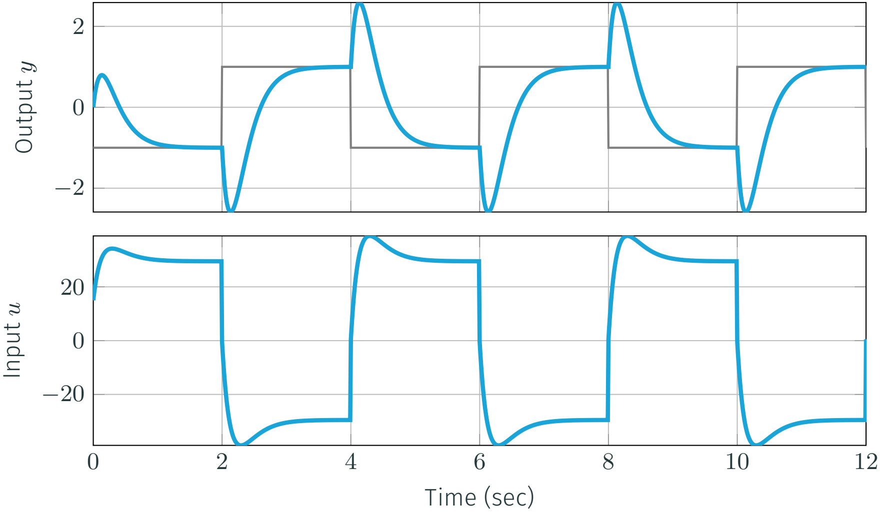
Reference computation

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$$

$$\bar{N} = (N_u + KN_x) = -0.5 + \begin{bmatrix} 14 & 57 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = -15$$

Implement the control law

$$u = -Kx + \bar{N}r = -\begin{bmatrix} 14 & 57 \end{bmatrix} x - 15r$$



Selection of Good Pole Locations

What are Good Pole Locations?

There are many ways to do this depending on the goals and the system. e.g.,

1. Place dominant second-order poles
   - Choose location for the ‘main’ behaviour, and damp the rest of the modes quickly
2. Model matching
   - Choose from a parameterized prototype response
3. Optimal control - Linear Quadratic Regulator
   - Define a ‘cost function’ and select poles to minimize it

Pole selection is often an *iterative scheme* before finding the best location.

We will cover the first two now, and return to the third later.

Dominant Second Order Poles

Idea: Chose the closed-loop system to have an *almost* second-order response

- Use time-domain specifications to locate ***dominant poles***

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

  using e.g., overshoot, settling time, etc
- Place the remaining poles so that they are 'much faster'
  - e.g., keep damped frequency ω_d and move real part to be $2\times$-$3\times$ faster than dominant poles

Some principles to keep in mind in order to minimize control effort

- It takes more control effort the farther poles are moved
- Moving almost uncontrollable modes is more difficult

Example - Placement of Dominant Mode

Design a state-feedback control law so the closed-loop system has no more than a 5% overshoot and a settling time less than 10 seconds.

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -0.1 & -0.35 & 0.1 & 0.1 & 0.75 \\ 0 & 0 & 0 & 2 & 0 \\ 0.4 & 0.4 & -0.4 & -1.4 & 0 \\ 0 & -0.03 & 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Example - Choose Target Model

Select second-order poles for 5% overshoot and a rise time less than 4 seconds.

Percent overshoot less than 5%

$$P.O. := M_p \times 100\% = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\zeta \geq -\frac{\ln M_p}{\sqrt{\ln(M_p)^2 + \pi^2}} = 0.69$$

Choose $\zeta = 0.7$

Settling time less than 10 sec

Time to settle to within $\delta = 1\%$ percent of the steady-state value.

$$T_s = \frac{-\log\delta}{\zeta\omega_n} = \frac{4.6}{\zeta\omega_n} = \frac{4.6}{\sigma}$$

$$10 \geq \frac{4.6}{\zeta\omega_n}$$

$$\omega_n \geq \frac{4.6}{10}\frac{1}{0.7} = 0.66$$

Choose $\omega_n = 0.7$

Example - Choose Target Model

Idea: Place remaining three poles faster than the dominant mode.

Natural frequency of dominant mode is $\omega_n = 0.7$.
Choose remaining poles approx $4\times$ faster. (rule of thumb)

Desired poles are now

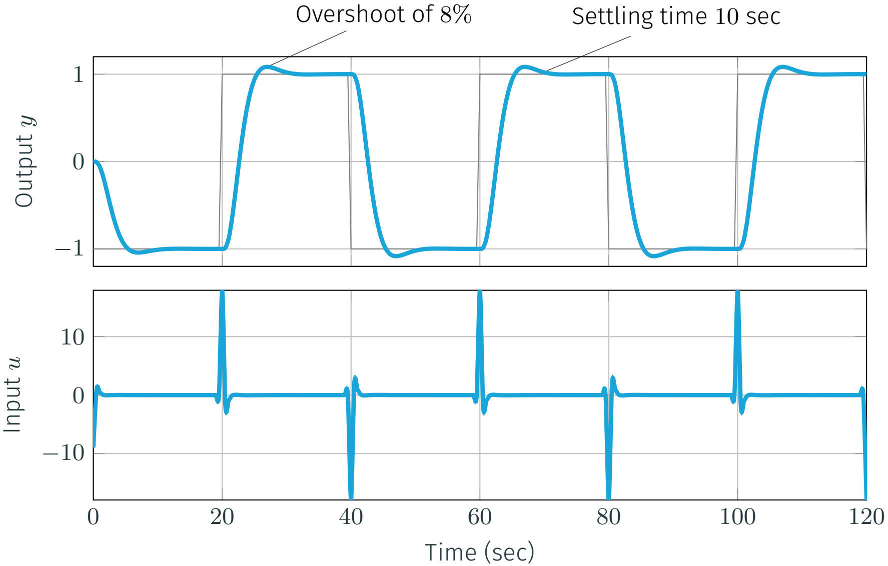
$$\begin{aligned} P &= \omega_n \begin{bmatrix} -\zeta + i\sqrt{1-\zeta^2} & -\zeta - i\sqrt{1-\zeta^2} & -4 & -4 & -4 \end{bmatrix} \\ &= 0.7 \begin{bmatrix} -0.707 + i0.707 & -0.707 - i0.707 & -4 & -4 & -4 \end{bmatrix} \end{aligned}$$

```
A = [0 2 0 0 0; -0.1 -0.35 0.1 0.1 0.75; 0 0 0 2 0; 0.4 0.4 -0.4 -1.4 0; ...
     0 -0.03 0 0 -1];
B = [0;0;0;0;1];
C = [0 0 1 0 0];
D = 0;

wn = 0.7;
zeta = 0.707;
P = wn * [roots([1 2*zeta 1]); -4*ones(3,1)];

K = acker(A,B,P);
```

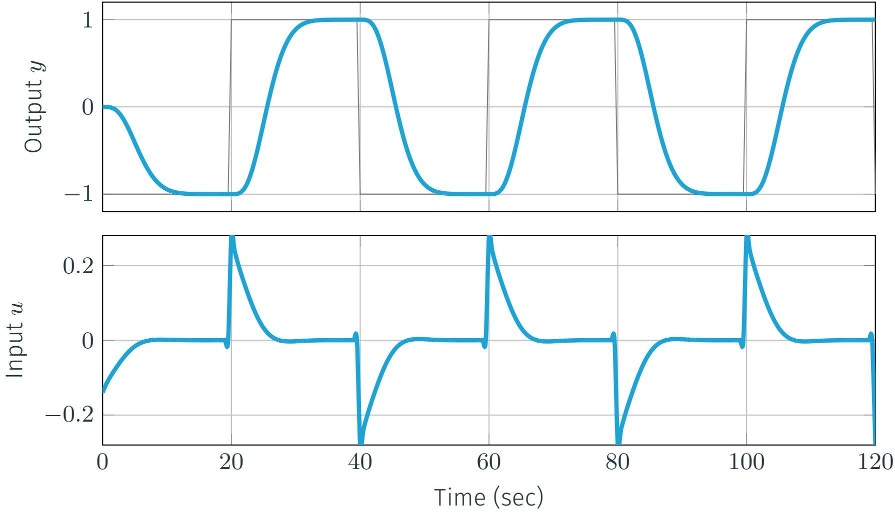
Example



Overshoot of 8%
Settling time 10 sec
Output y
1
0
−1
Input u
10
0
−10
0
20
40
60
80
100
120
Time (sec)

Example - Slower Non-dominant Poles

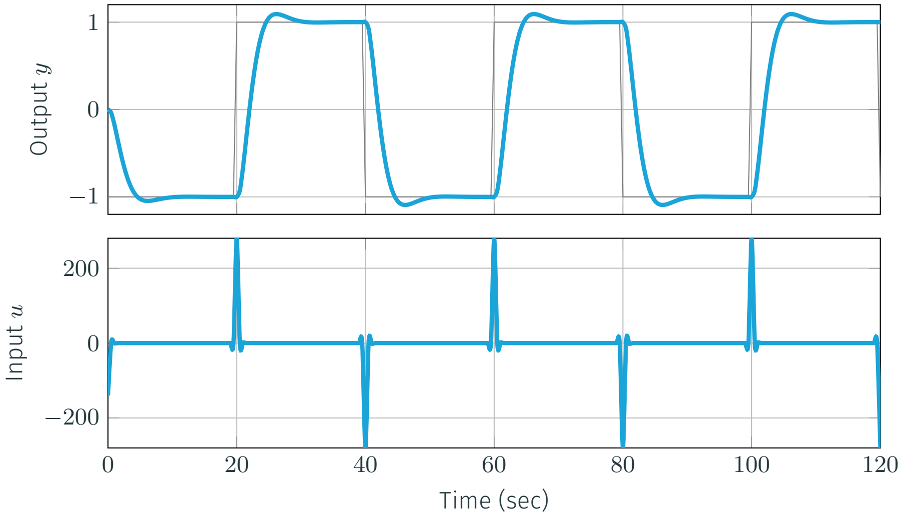
Place non-dominant poles at -1



Non-dominant poles influence behaviour. Settling time slower.

Example - Faster Non-dominant Poles

Place non-dominant poles at -10



Gain is extremely high to move non-dominant poles to a high frequency.

Idea: Select characteristic equation that is known to give a good response

For example, the reverse Bessel polynomials are given by:

$$\theta_n(s) = \sum_{k=0}^{n} a_k s^k$$

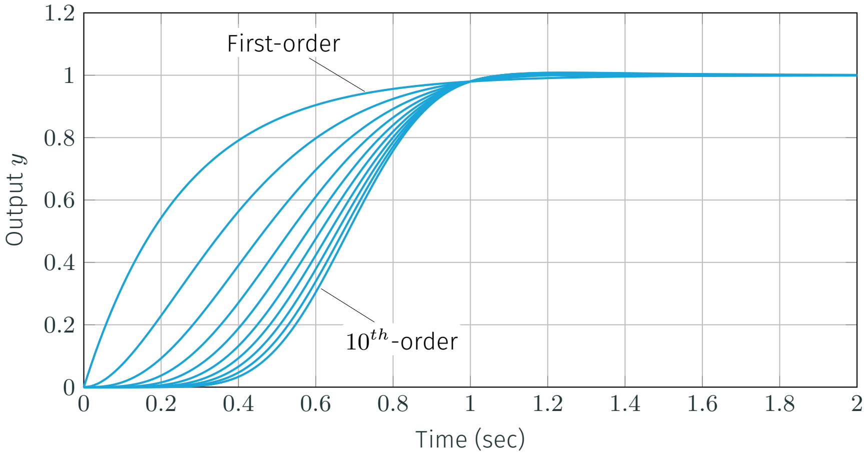
where

$$a_k = \frac{(2n-k)!}{2^{n-k} k! (n-k)!} \quad k = 0, 1, \ldots, n$$

$$\begin{aligned}
n &= 1 & \theta_1(s) &= s + 1 \\
n &= 2 & \theta_2(s) &= s^2 + 3s + 3 \\
n &= 3 & \theta_3(s) &= s^3 + 6s^2 + 15s + 15 \\
n &= 4 & \theta_4(s) &= s^4 + 10s^3 + 45s^2 + 105s + 105 \\
n &= 5 & \theta_5(s) &= s^5 + 15s^4 + 105s^3 + 420s^2 + 945s + 945
\end{aligned}$$

Response of Bessel Filter

Step response of bessel filters



Example - Bessel Filter

Design a state-feedback control law so the the closed-loop system has no more than a 5% overshoot and a settling time less than 10 seconds.

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -0.1 & -0.35 & 0.1 & 0.1 & 0.75 \\ 0 & 0 & 0 & 2 & 0 \\ 0.4 & 0.4 & -0.4 & -1.4 & 0 \\ 0 & -0.03 & 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Fifth-order reverse Bessel function has poles

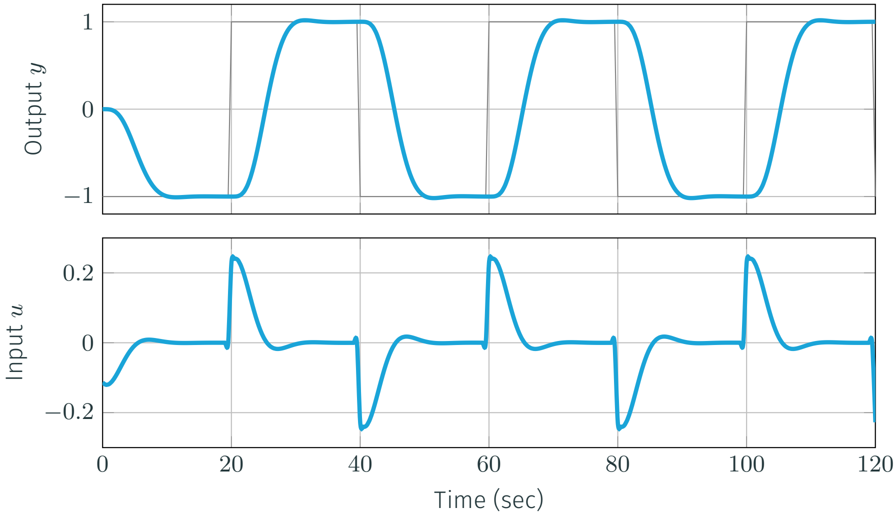
$$P =$$

$$\begin{bmatrix} -0.5906 + 0.9072i & -0.5906 - 0.9072i & -0.8516 + 0.4427i & -0.8516 - 0.4427i & -0.9264 \end{bmatrix}$$

Use place to place the poles at $P/T_s = P/10$:

$$K = \begin{bmatrix} 0.1571 & 0.2234 & -0.0434 & 0.0345 & -0.1912 \end{bmatrix}$$

Example - Bessel Filter



Easy to tune and good response.

Good Pole Locations - Summary

There are many ways to do this depending on the goals and the system. e.g.,

1. Place dominant second-order poles
   - Choose location for the 'main' behaviour, and damp the rest of the modes quickly
2. Model matching
   - Choose from a parameterized prototype response
3. Optimal control - Linear Quadratic Regulator
   - Define a 'cost function' and select poles to minimize it

Pole selection is often an *iterative scheme* before finding the best location.

We will return to LQR control later on.